International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Qualifying round

Solutions for grade R5

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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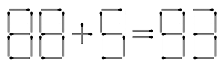
1. Is it possible to place distinct one-digit numbers in a 2×4 rectangle so that the sum of any two numbers which are adjacent by side is a prime number?
Remark. We remind that a prime number is an integer greater than 1 that is divisible only by 1 and itself. (*M. Karlukova*)

Answer: yes, for example, like this:

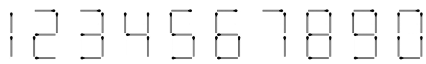
7	4	3	8
6	1	2	5

Criteria. Just an example – 7 points (number 0 can be used too). Answer "yes" with incorrect example or without example – 0 points.

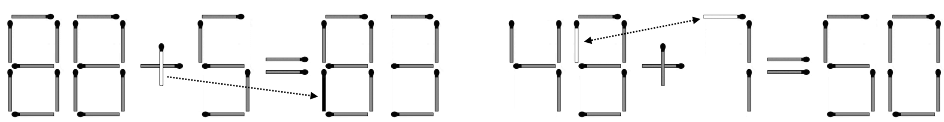
2. First grader Paul laid out the equation $88+5=93$ using matchsticks (see on the right). The teacher gave him an 'excellent' grade.
(a) Help first grader Andrew also get an 'excellent' grade by moving exactly one matchstick in Paul's example so that the equation remains valid.



(b) Come up with your own valid example using 5 different digits laid out with matchsticks, so that one matchstick can be moved from one digit to another to obtain a valid equation again (in other words, the arithmetic operation signs cannot be changed, and the sign "=" cannot be crossed out). Samples of all digits are shown on the right. (*P. Mulenko*)

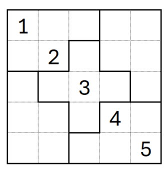


Answer: a) $88-5=83$; b) for example, $43+7=50$ transforms into $49+1=50$.



Criteria. Only item a) — 2 points; only item b) — 5 points.

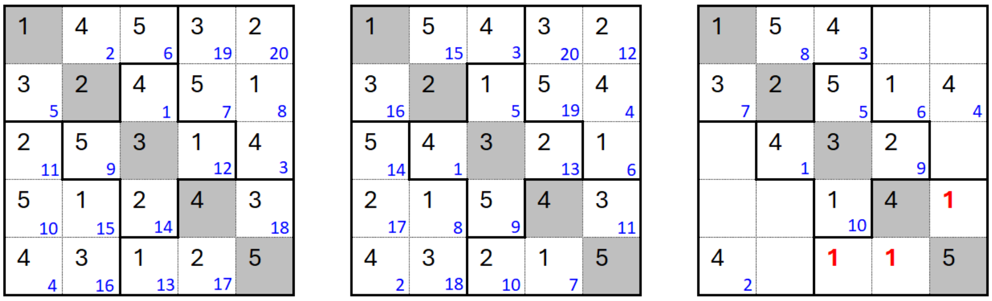
3. How many solutions does the figure sudoku shown on the right have? Don't forget not only to find all the solutions but to explain that you have found all of them.
Remark. According to the rules of a 5×5 figure sudoku, all digits from 1 to 5 must appear in each row, column, and each highlighted block. (*M. Karlukova*)



Answer: 2 solutions.

Solution. In the central cross, the number 4 can be either in the left or in the upper empty cell. If you put 4 in the upper cell, then the rest of the sudoku can be solved unambiguously (see the picture on the left, the order of placing the numbers is indicated in the lower right corner of the cells).

If you put 4 in the left cell, then after placing all the fours, it is not possible to restore the next numbers unambiguously. Therefore, let's consider two possible options of the number in the upper cell of the central cross (1 or 5). If there is 1 there, then the rest of the sudoku can be solved unambiguously (see the central picture); if there is 5 there, then one way or another a contradiction is obtained (see the picture on the right: in the lower right figure there is no place to put 1).



Criteria. Both correct solutions are indicated without justification that there are no other solutions – 2 points.

4. Connor and Mary take turns placing the signs + and − in the squares of a chessboard (Connor starts) according to the following rules.
   - Each turn, the player chooses any free square and places one sign of their choice in it.
   - If, after a player's move, there is an equal number of pluses and minuses in squares of the chosen color, that player automatically loses.
   - After the board is filled, it is determined (for black and white squares separately) which signs are in greater number. If one color has more pluses and the other has more minuses, then Mary wins; otherwise, Connor wins.

   Who can ensure their victory, and how should they act to win? (*P. Mulenko*)

   Answer: Mary can ensure her victory by placing a sign opposite to Connor's sign in a cell of the opposite color on her first move.

   Solution. Let's say Connor places + in a black cell on her first move. This means that there will always be more pluses than minuses in black cells (to change this, one of the players would have to equalize the number of signs first, which is automatically considered a loss). Thus, Mary only needs to place − in a white cell on her first move to ensure that there are more minuses in white. The same reasoning applies to the other three options for Connor's move (+ in white, − in black, − in white).

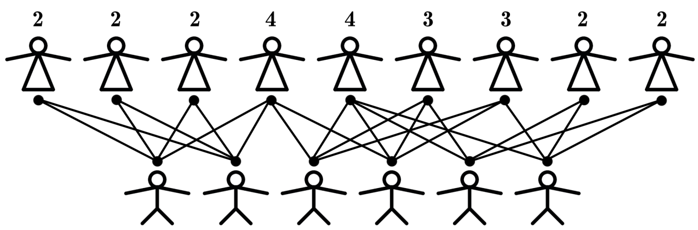
   Criteria. Only the correct answer (Mary) – 0 points. Mary's first move is indicated correctly, but it is not explained why it leads to victory – 2 points. For the remark that the result for each of the colors is determined by the first sign placed there, 2 points are awarded.

5. Several children came to participate in the final stage of an Olympiad.
   - Among them, there are two girls, each of whom knows exactly four boys.
   - There are five girls, each of whom knows exactly two boys.
   - Each of the remaining girls (if any) knows exactly three boys.
   - Each boy knows four girls.

   What is the minimum number of girls who could have come to the Olympiad? Don't forget to prove that this is indeed the minimum. (*L. Koreshkova*)

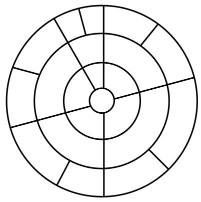
   Answer: 9 girls.

   Solution. If x is the number of girls with three acquaintances, then the total number of acquaintances is $2 \cdot 4 + 5 \cdot 2 + 3x = 18 + 3x$. Since each boy knows 4 girls, then the number of acquaintances must be divisible by 4. The minimum x for which this is true is 2. The total number of girls is not less than $2 + 5 + 2 = 9$. An example for 9 girls is shown below.



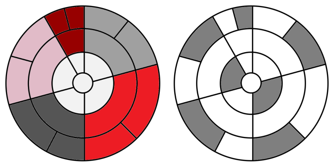
Criteria. Example only – 2 points. Assessment only – 3 points. Answer only (9) – 0 points.

6. The company's office is located on a separate floor of a business center (see the figure). Each room contains a department managed by a manager. The company director decided to promote some department heads to chief managers. However, it is not allowed for two neighboring departments to be managed by chief managers (otherwise, they will argue through the wall about who is more important). What is the maximum number of managers that can be promoted? Don't forget to explain why this number is indeed maximum. (*I. Tumanova*)



Answer: 7 managers.

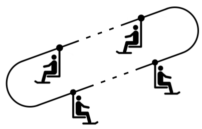
Solution. Let's divide the office into 6 zones (shown in different colors in the picture on the left). In the central zone there are no more than two chief managers (if the central one was promoted, then the others cannot be promoted; if not, then a maximum of two can be chosen from among his 5 neighbors). And in each of the other 5 zones there is no more than one chief manager. In total, no more than 7 managers can be promoted.



An example of how to promote 7 managers is shown on the right.

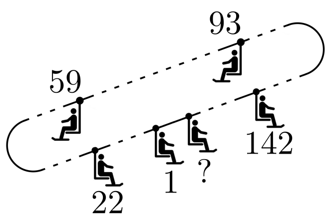
Criteria. Only the answer (7) – 0 points. The answer with an example, but without an estimation – 2 points. If some cases are omitted in the estimation (for example, only cases are considered when the manager in the central room is not promoted) – no more than 5 points. "Greedy" reasoning (e. g. "to maximize the result, the outer ring should have as many top managers as possible") is incorrect.

7. The chairs on the ski lift are numbered sequentially: 1, 2, 3, etc. The distances between each pair of adjacent chairs are the same. During a thunderstorm, the ski lift stopped, and at that moment chair 22 was at the same height as chair 59, while chair 93 was at the same height as chair 142. Determine the number of chairs on the ski lift. (*L. Koreshkova*)



Answer: 154.

Solution.
Note that seats 22 and 142 should be on one side, and 59 and 93 on the other side (indeed, if 22 and 93 were on one side, and 59 and 142 on the other, the order of the seats would be violated). This means that there is the same amount of seats between 142 and 22 as between 59 and 93, i.e. $93 - 59 - 1 = 33$ seats. 21 of them are numbered from 1 to 21, which means that 12 seats have numbers greater than 142. It turns out that there are $142 + 12 = 154$ seats in total.



Criteria. The correct answer (154) is given without any justification – 1 point.
The solution is correct, but the fact that 22 and 142 are on the same side is not justified in any way (it is just drawn without comment) – penalty of 1 point. Errors in ± 1 when calculating the distances between the chairs, affecting the answer – penalty of 2 points. Arithmetic errors – penalty of 1 point.

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Solutions for grade R6

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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1. Is it possible to place all numbers from 1 to 8 in a 2×4 rectangle so that the sum of any two numbers which are adjacent by side is a prime number?
   Remark. We remind that a prime number is an integer greater than 1 that is divisible only by 1 and itself. (*M. Karlukova*)

   Answer: yes, for example, like this:

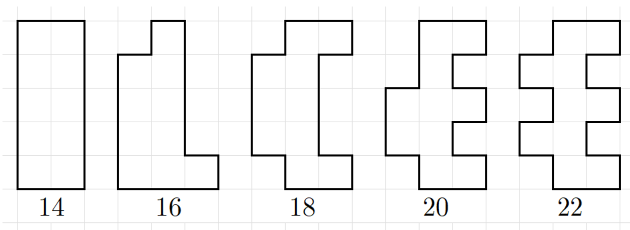
7	4	3	8
6	1	2	5

   Criteria. Just an example – 7 points. Answer "yes" with an incorrect example or without an example- 0 points. The number 0, unlike the problem for grade R5, cannot be used.

2. In independent work, the teacher gave each of the 10 students a polygon drawn on grid paper along its lines, with an area of 10 cells, and asked them to calculate its perimeter (each student received their own polygon). It turned out that all students had different answers ranging from 13 to 22. What is the minimum number of incorrect answers among them? (*S. Pavlov*)

   Answer: 5 (all odd answers).

   Solution. Let's prove that the perimeter is always even. Indeed, let's trace the figure along the perimeter, returning to the starting point. Since the drawing will end at the same vertex where it began, the number of unit segments drawn upward is equal to the number of unit segments drawn downward. This means that the total number of vertical segments is even. The same is true for horizontal segments. So, such a polygon has an even perimeter, so all odd answers are incorrect.
   For even perimeters, examples are constructed:



   Criteria. 1 point if it is mentioned that all even results can be obtained, but odd ones cannot (or (in)correct answers are listed directly).
   Another 3 points – examples (5 correct examples – 3 points, 4 examples – 2 points, 3 examples – 1 point, less – 0 points).
   Another 3 points – proof that odd perimeters are impossible.

3. Connor and Mary take turns placing the signs $+$ and $-$ in the squares of an 8×8 board (Connor starts) according to the following rules.
   - Before each move, the player chooses a sign and a color (red, green, or blue) and writes the chosen sign in any free square on the board.
   - If, after a player's move, there is an equal number of pluses and minuses of the chosen color, that player automatically loses.
   - After the board is filled, for each color, it is determined which signs were in greater number. If there are more minuses in one or all three colors, then Mary wins; otherwise, Connor wins.

Who can ensure their victory, and how should they act to win? (*P. Mulenko*)

Answer: Mary can ensure her victory.

Strategy. Let's say Connor's first move was + in blue. This means that there will always be more pluses than minuses among the blue signs (to change this, one of the players would have to equalize the number of signs first, which is automatically considered a loss). Therefore, Mary has no choice but to repeat Connor's moves in the same color until Connor decides to place a sign *in the second color*, after which she will have to place a sign *in the third color* so that in the end there will be more minuses in one or three colors. After that, the course of the game does not matter, Mary only needs to make sure that her moves do not equalize the quantity of signs in any of the colors.

The only exception is if Connor never places a sign in the second color. In this situation, Mary will have to make her last move by placing a sign in the second color so that exactly one of the two colors has more minuses.

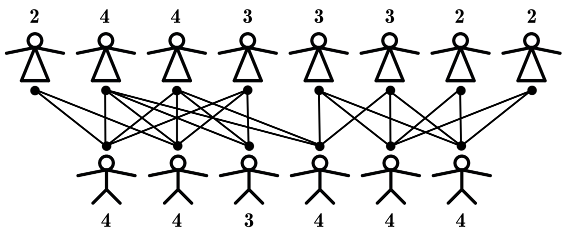
Criteria. For noticing that the result for each color is determined by the first sign of that color, 2 points are awarded. For the strategy in an exceptional case, 1 point is awarded. If the strategy is described, but the exception is forgotten, then no more than 5 points are awarded for the problem.

4. Several children came to participate in the final stage of an Olympiad.
   - Among them, there are two girls, each of whom knows exactly four boys.
   - There are three girls, each of whom knows exactly three boys.
   - Each of the remaining girls (if any) knows exactly two boys.
   - No boy knows more than four girls.

   What is the maximum number of girls who could have come to the Olympiad if a total of 15 children participated? Don't forget to prove that this is indeed the maximum number. (*L. Koreshkova*)

   Answer: 8 girls.

   Solution. Let x be the number of girls who know two boys each; then there are $15-2-3-x=10-x$ boys. Then the number of acquaintances does not exceed $4\cdot(10-x)=40-4x$ (since each boy knows no more than 4 girls), and at the same time, from the girls' point of view, it is equal to $2\cdot 4+3\cdot 3+x\cdot 2=17+2x$. That is, $17+2x\leq 40-4x$, whence $x\leq 3$. So not more than $2+3+3=8$ girls could come. An example for 8 girls is shown below.



   Criteria. Example only – 2 points. Assessment only – 3 points. Answer only (8) – 0 points.

5. Consider any three-digit number denoted as $\overline{fdi}$, where each letter represents a separate digit. How many numbers $\overline{fdi}$ exist such that $\overline{idf}$ is divisible by $\overline{fdi}$? (*L. Koreshkova*)

   Answer: 90.

   Solution. If $\overline{idf}\,\vdots\,\overline{fdi}$, then also $(\overline{idf}-\overline{fdi})\,\vdots\,\overline{fdi}$, $\overline{idf}-\overline{fdi}=(100i+10d+f)-(100f+10d+i)=99i-99f=99(i-f)$, thus $99(i-f)=(9\cdot 11\cdot(i-f))\,\vdots\,\overline{fdi}$.

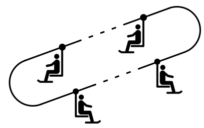
   This is obviously true for $f=i$, since then $\overline{fdi}=\overline{idf}$. There are 90 such numbers (9 options for the digit $f=i$, since it cannot be zero, and 10 options for the digit d).

   If $f\neq i$, then $f-i\geq 2$ (otherwise $99(i-f)\leq 100$) and $f-i\leq 9-1=8$, from which we can directly verify that no suitable numbers can be found:
   - $i-f=2$: $99(i-f)=198$, but 891 is not divisible by 198;
   - $i-f=3$: $99(i-f)=297$, but 792 is not divisible by 297;
   - $i-f=4$: $99(i-f)=396\,\vdots\,396, 198, 132$, but none of them fit;
   - $i-f=5$: $99(i-f)=495\,\vdots\,495, 165$, but none of them fit;

- $i - f = 6$: $99(i-f) = 594 \vdots 594, 297, 198$, but none of them fit;
- $i - f = 7$: $99(i-f) = 693 \vdots 693, 231$, but none of them fit;
- $i - f = 8$: $99(i-f) = 792 \vdots 792, 396, 264, 198, 132$, but none of them fit.

Criteria. 1 point for indicating that all numbers of the form $\overline{fdf}$ are suitable, and another 1 point for correctly indicating their number (90). 5 points for proving that there are no other numbers.

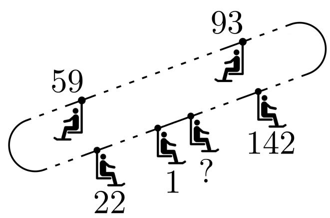
6. The chairs on the ski lift are numbered sequentially: 1, 2, 3, etc. The distances between each pair of adjacent chairs are the same. During a thunderstorm, the ski lift stopped, and at that moment chair 22 was at the same height as chair 59, while chair 93 was at the same height as chair 142. Determine the number of chairs on the ski lift. (*L. Koreshkova*)



Answer: 154.

Solution.

Note that seats 22 and 142 should be on one side, and 59 and 93 on the other side (indeed, if 22 and 93 were on one side, and 59 and 142 on the other, the order of the seats would be violated). This means that there is the same amount of seats between 142 and 22 as between 59 and 93, i.e. $93 - 59 - 1 = 33$ seats. 21 of them are numbered from 1 to 21, which means that 12 seats have numbers greater than 142. It turns out that there are $142 + 12 = 154$ seats in total.



Criteria. The correct answer (154) is given without any justification – 1 point.
The solution is correct, but the fact that 22 and 142 are on the same side is not justified in any way (it is just drawn without comment) – penalty of 1 point. Errors in ± 1 when calculating the distances between the chairs, affecting the answer – penalty of 2 points. Arithmetic errors – penalty of 1 point.

7. Sophie has seven friends: Alice, Bella, Dana, Grace, Helena, Jenna, Vicky. Their photos (a total of 7 pictures – one for each friend) are lying in two stacks in an unknown arbitrary order. In one move, Sophie takes several (one or more) consecutive photos from the top of any stack and places them on top of the other stack without changing the order. Is Sophie always able to arrange the photos into one stack with alphabetical order of friends' names (listing from the bottom up) after no more than 13 moves? (*S. Pavlov*)

Answer: yes.

Solution. On the first move, we make one stack out of two (let's call it the *left* one).
On the second move, we relocate Alice's photo and all photos above it to the right stack. After 2 moves, Alice's photo ended up at the bottom of the right stack.
On the third move, we relocate everything from the right stack, except Alice's photo, to the left stack.
On the fourth move, we relocate Bella's photo and all photos above it to the right stack. As a result, after the 4th move, Alice's and Bella's photos are in place (at the bottom of the right stack).
We continue to act in this way, spending two moves for each subsequent photo. (Perhaps, in some cases, we won't have to move anything, then we'll save moves.) After no more than 12 moves, it turns out that the bottom of the right stack contains 6 photos in the right order. If Vicky's photo is in the left stack, then the last (13th) move will transfer it to the right one, and if it is in the right stack, then it is already in the right place (at the top).

Criteria. Only the answer – 0 points. Any reasoning leading to the answer "no" is worth 0 points.
The idea of putting the first photo in place in two moves – 3 points. Forgot to put Vicky's photo back in place – a penalty of 1 point.

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Solutions for grade R7

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1. Is it possible to place all numbers from 1 to 9 in a 3×3 square so that the sum of any two numbers which are adjacent by side is a prime number? (*M. Karlukova*)

   Answer: no.

   Solution. The adjacent cells must contain numbers of different parity (otherwise they add up to an even number greater than 2, i.e. they must be composite). That is, the central cell must contain an odd number that will add up to all four even numbers, but for any odd digit at least one of the sums will be composite: $1 + 8 = 3 + 6 = 5 + 4 = 7 + 2 = 9$, $9 + 6 = 15$.

   Criteria. For noticing that adjacent cells contain numbers of different parity, 2 points are awarded.

2. Kate drew squares with even sides ranging from 2 cells to 2024 cells in her notebook. Ivy drew a rectangle under each of Kate's squares with the same perimeter but with a width that is 1 less than that of the square. Which girl has a greater total area (in cells) and by how much? (*P. Mulenko*)

   Answer: Kate's area is 1012 cells greater.

   Solution. Let's consider Kate's arbitrary square with side x. Then Ivy drew a rectangle under it with sides $x-1$ and $x+1$ (so that the perimeter does not change), whose area is $(x-1)(x+1) = x^2-1$. Thus, each of Ivy's rectangles has an area 1 less than Kate's square. And since Kate drew $2024 : 2 = 1012$ squares, her total area is 1012 cells greater.

   Criteria. It is shown that the area of the corresponding square is 1 more than the area of the rectangle – 5 points.

3. Find all quadruples of distinct digits $a < b < c < d$ such that $\overline{ab} \cdot \overline{dc} = \overline{ba} \cdot \overline{cd}$.
   Remark. The notation $\overline{ab}$ denotes a two-digit number composed of the digits a and b. (*L. Koreshkova*)

   Answer: there are 5 such quadruples: (1, 2, 3, 6); (1, 2, 4, 8); (2, 3, 4, 6); (2, 3, 6, 9); (3, 4, 6, 8).

   Solution. $\overline{ab} \cdot \overline{dc} = (10a+b)(10d+c) = 100ad+10ac+10bd+bc$; similarly $\overline{ba} \cdot \overline{cd} = 100bc+10bd+10ac+ad$. Equating these expressions, we obtain $a \cdot d = b \cdot c$.
   - The digits definitely do not include 0, 5, or 7 (otherwise, two pairs cannot be constructed from different digits).
   - If the smallest digit (that is, a) is 1, then the following options are suitable: $1 \cdot 6 = 2 \cdot 3$ and $1 \cdot 8 = 2 \cdot 4$, so we have two groups: (1, 2, 3, 6) and (1, 2, 4, 8).
   - If $a = 2$, then there are two more groups: (2, 3, 4, 6) and (2, 3, 6, 9).
   - If $a = 3$, then there is one more group: (3, 4, 6, 8).

   There are no further groups, that is, there are only 5 quadruples.

   Criteria. It is shown that there are no 0, 5 and 7 among the numbers – 3 points. For any lost fundamental case during the enumeration, 2 points are subtracted. With a fully described enumeration system and consideration of all fundamental cases, 1 point is subtracted if the options within a case is lost.

4. Connor and Mary take turns placing the numbers $+1$ and -1 in the squares of an 8×8 board (Connor starts) according to the following rules.
   - Before each move, the player chooses a number and a color (red, green, or blue) and writes the chosen number in any free square on the board.
   - If, after a player's move, the sum of the numbers of the chosen color equals zero, that player automatically loses.

- After the board is filled, the sums of the numbers of each color are calculated. If the product of these sums is positive, Connor wins; if negative, Mary wins; if any color was never used, it is not included in the count.

Who can ensure their victory, and how should they act to win? (*P. Mulenko*)

Answer: Mary can ensure her victory.

Strategy. Let's say Connor's first move was $+1$ in blue. This means that the sum of the blue numbers will always be positive (to change this, one of the players would have to make it equal to zero first, which is automatically considered a loss). So Mary has no choice but to repeat Connor's moves in the same color until Connor decides to place a number *in the second color*, after which she will have to place a number *in the third color* so that the sum in one or three colors ends up being negative (which will ensure a negative product). After that, the course of the game does not matter, Mary only needs to make sure that her moves do not make the sum in any of the colors equal to zero.

The only exception is if Connor never places a number in the second color. In this situation, Mary will have to put the number in the second color as her last move so that the sum is negative in exactly one of the two colors.

If Connor put -1 as his first move, Mary does the same: when adding the second color, she adds the third with the right sign, otherwise, she puts $+1$ in a different color as her last move.

Criteria. For noticing that the sum of numbers of the same color is determined by the first addend, 2 points are given. For the strategy in an exceptional case, 1 point is given. If the strategy is described, but the exception is forgotten, then no more than 5 points are given for the problem.

5. Sophie has ten friends: Alice, Chloe, Diana, Fiona, Inna, Jenna, Karina, Lily, Mary, Olivia. Their photos (a total of 10 pictures- one for each friend) are lying in two stacks in an unknown arbitrary order. In one move, Sophie takes several (one or more) consecutive photos from the top of any stack and places them on top of the other stack without changing the order. Is Sophie always able to arrange the photos into one stack with alphabetical order of friends' names (listing from the bottom up) after no more than 21 moves? (*S. Pavlov*)

Answer: yes; in fact, even 19 moves are enough.

Solution. On the first move, we make one stack out of two (let's call it the *left* one).

On the second move, we relocate Alice's photo and all photos above it to the right stack. After 2 moves, Alice's photo ended up at the bottom of the right stack.

On the third move, we relocate everything from the right stack, except Alice's photo, to the left stack.

On the fourth move, we relocate Chloe's photo and all photos above it to the right stack. As a result, after the 4th move, Alice's and Chloe's photos are in place (at the bottom of the right stack).

We continue to act in this way, spending two moves for each subsequent photo. (Perhaps, in some cases, we won't have to move anything, then we'll save moves.) After no more than 18 moves, it turns out that the bottom of the right stack contains 9 photos in the right order. If Olivia's photo is in the left stack, then the last (19th) move will transfer it to the right one, and if it is in the right stack, then it is already in the right place (at the top).

Criteria. Only the answer – 0 points. Any reasoning leading to the answer "no" is worth 0 points.

The idea of putting the first photo in place in two moves – 3 points. Forgot to put Olivia's photo back in place – a penalty of 1 point.

6. Several children came to participate in the final stage of an Olympiad.
   - Among them, there are two girls, each of whom knows exactly four boys.
   - There are three girls, each of whom knows exactly three boys.
   - Each of the remaining girls (if any) knows exactly two boys.
   - No two children of the same gender know each other.
   - If two children know each other, each of them can give a cheat sheet to the other one.

   It turned out that any girl could send a cheat sheet to any boy (even one she doesn't know, through other children), but if the organizers start closely monitoring at least one pair of children who know each other, this possibility would break down (meaning there would be some boy and girl who would no longer be able to send a cheat sheet to each other). Who came to the Olympiad in a greater number, boys or girls, and by how many? (*L. Koreshkova, P. Mulenko*)

Answer: there are 8 more boys than girls.

Solution. Let us denote the number of girls with two acquaintances as x, then the total number of acquaintances is $2 \cdot 4 + 3 \cdot 3 + x \cdot 2 = 17 + 2x$. On the other hand, if we draw a graph describing acquaintances (where children are vertices and acquaintances are edges), then this graph will turn out to be a tree. Indeed, the graph was initially connected, since each girl could pass the cheat sheet to any boy (and, via him, to any other girl), but deleting any edge makes it disconnected, which is one of the definitions of a tree. Therefore, the number of edges (i. e. acquaintances) is 1 less than the number of vertices: $17+2x = 2+3+x+m-1$, where m is the number of boys who came to the Olympiad. Thus, $m = x + 13 = (x + 5) + 8$, where $x + 5$ is the total number of girls who came to the Olympiad, that is, 8 more boys came.

Criteria. The remark that there is 1 more children than acquaintances is worth 3 points. If calculations show that there are 13 more boys, due to forgotten girls with 3 and 4 acquaintances, we give 5 points. 2 points if the number of acquaintances is expressed through the number of girls.

A solution of the form "let's draw 5 girls and 13 boys, and then add +1D, +1M" – 2 points.

Only the correct answer with an example – 1 point.

7. Two vending machines sell the same burger, but each of them is broken and changes all the numbers on the screen by some constant value (all other information is correct).

The company servicing these machines decided to display appropriate notifications during repairs. The following appeared on the screens:

The other machine displays all numbers on the screen 2 more than they actually are. Burger: **$2**	*The other machine displays all numbers on the screen 8 less than they actually are.* Burger: **$10**

What is the real price of the burger? (*P. Mulenko*)

Answer: $5.

Solution. From the displayed texts it follows that the first machine decreases all numbers by some value (let it be x), and the second machine, on the contrary, increases all numbers by some value (let it be y). Then the first machine outputs the real increase of the second machine y, decreased by x, and we get 2; and the second machine outputs the real decrease of the first machine x, increased by y, and we get 8:

$$\begin{cases} y - x = 2, \\ x + y = 8, \end{cases} \quad \Rightarrow \quad \begin{cases} y = 5, \\ x = 3. \end{cases}$$

Thus, in reality, the burgercosts $2 + x = 10 - y = 2 + 3 = 10 - 5 = 5$ dollars.

Criteria. It is calculated how much exactly one of the machines changes the value, without finding the final answer – 6 points. A system of equations or an equation leading to the correct answer is correctly composed but not solved – 4 points.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Qualifying round

Solutions for grade R8

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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1. The chess piece "*knishop*" moves one move like a bishop, the next move like a knight, then again like a bishop, and so on (in other words, the moves of bishop and knight alternate). Can it traverse the chessboard visiting all 64 squares exactly once? (*O. Pyayve*)

Answer: yes. An example is shown below: the knishop starts on square number 1 with a bishop move to square 2, then a knight to square 3, etc. (the squares it gets to with a bishop move are marked in bold). Note that in this example, each quadrant of the board is traversed in the same order.

8	**52**	**58**	**64**	49	**36**	45	47	**38**
7	61	**56**	53	59	**42**	**40**	**34**	**44**
6	63	**50**	55	57	**48**	37	39	**46**
5	**54**	**60**	**62**	51	33	43	41	35
4	**20**	**26**	**32**	17	**4**	13	15	**6**
3	29	**24**	21	27	**10**	**8**	**2**	**12**
2	31	**18**	23	25	**16**	5	7	**14**
1	**22**	**28**	**30**	19	1	11	9	3
	a	b	c	d	e	f	g	h

Criteria. Example — 7 points.

The reasoning "yes, because the knight changes the color of the square with each move, but the bishop does not" – 2 points. If the solution is given for only one quadrant – 4 points.

2. A construction set consists of white cubes. Paul assembles a large cube from all the cubes, then selects 4 faces of the large cube and paints them red. After that, he disassembles the large cube and counts the cubes that have at least one face painted red. Paul ends up with 431 such cubes. Could this happen? If so, find all possible total numbers of cubes. (*L. Koreshkova*)

Answer: Yes, it could, if Paul initially had $11^3 = 1331$ cubes.

Solution. Let an edge of a large cube consist of n edges of small cubes. There are two ways to paint 4 faces of a cube: a "ring", leaving two opposite faces unpainted, and a "shelf", leaving two adjacent faces unpainted.

In a "ring", the total number of painted cubes is equal to the number of cubes in the 4 faces minus the number of cubes in the 4 edges, which were counted twice, that is, $4n^2 - 4n$.

In a "shelf", the total number of painted cubes is equal to the number of cubes in the 4 faces minus the number of cubes in the 5 common edges plus 2 vertices of the cube, where the triplets of faces and edges meet (since we have previously counted them 3 times in the faces and subtracted them 3 times in the edges), that is, $4n^2 - 5n + 2$.

Then we can act in different ways.

The first method. If $n \leq 10$, then Pasha painted less than $4n^2 \leq 400$ cubes, and if $n \geq 12$, then more than $4n^2 - 5n > 4n(n-2) \geq 4 \cdot 12 \cdot 10 = 480$ cubes. Thus, if Paul could have gotten 431 painted cubes, then only if $n = 11$. Indeed, if he painted a "shelf" he would have gotten $4 \cdot 11^2 - 5 \cdot 11 + 2 = 431$ painted cubes.

The second method. It is necessary to find positive integer roots of each of the two equations: $4(n^2 - n) = 431$ (for the "ring" coloring) and $4n^2 - 5n + 2 = 431$ (for the "shelf" coloring). Solving both equations using discriminants gives irrational solutions in the first equation and roots $\frac{5 \pm 83}{8}$ in the second, only one of which is integer (namely $n = \frac{5+83}{8} = 11$).

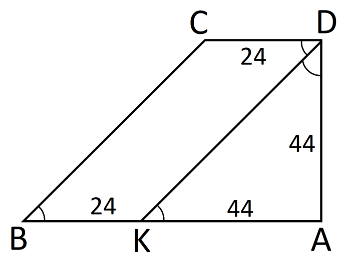
Criteria. 3 points if quadratic equations were obtained without further correct advances.

4 points if one of the two cases (a shelf or a ring) was completely calculated.

3. In a trapezoid $ABCD$, the base CD equals 24, $AD = 44$, and the angle B is half of the angle D. What is the maximum possible area of the trapezoid? (*L. Koreshkova, A. Tesler*)

Answer: 2024.

Solution. Let us draw the bisector DK of the angle D, where the point K lies on the base AB. Then $\angle CDK = \angle CBK$ by condition (that is, $DCBK$ is a parallelogram, that is, $BK = CD = 24$) and $\angle CDK = \angle DKA$ as they lie crosswise (that is, $\triangle ADK$ is isosceles, $AK = AD = 44$). The maximum area requires the maximum height, which is achieved when $\triangle ADK$ is right (in other words, the height of the trapezoid is certainly not greater than the lateral side). In total, the maximum area is equal to $\frac{24+68}{2} \cdot 44 = 2024$.



Criteria. Reasoning about the bisector and the isosceles triangle – 2 points.

If it is not proved that $DCBK$ is a parallelogram, – a penalty of 2 points.

4. Two vending machines sell the same burger, but each of them is broken and changes all the numbers on the screen by multiplying them by some constant value (all other information is correct).
The company servicing these machines decided to display appropriate notifications during repairs. The following appeared on the screens:

*The other machine displays all numbers on the screen **100%** more than they actually are.* Burger: **$2**	*The other machine displays all numbers on the screen **6** times less than they actually are.* Burger: **$12**

What is the real price of the burger? (*P. Mulenko*)

Answer: $4.

Solution. From the displayed texts it follows that the first machine reduces all numbers by some constant positive number of times (let it be x), and the second machine, on the contrary, increases all numbers by some constant number of times (let it be y). Then the first machine outputs the real increase of the second machine y as a percentage, reduced by x times, and we get 100%; and the second machine outputs the real decrease of the first machine x, increased by y times, and we get 6:

$$\begin{cases} \frac{(y-1)\cdot 100\%}{x} = 100\%, \\ x \cdot y = 6, \end{cases} \Rightarrow \begin{cases} y - 1 = x, \\ xy = 6, \end{cases} \Rightarrow \begin{cases} y = x + 1, \\ x(x+1) = 6. \end{cases}$$

The second equation can be solved as a quadratic. But you can also just guess the only positive root $x = 2$ (it is unique, since when the positive number x increases, the value $x(x+1)$ also increases). So, $x = 2$ and $y = 3$. Thus, a burger actually costs $2 \cdot x = 12 : y = 4$ dollars.

Criteria. An answer with verification that it meets all conditions – 2 points.

5. Sophie has N friends with different names: Amelia, Bianca, Eliza, ..., Yana. Their photos (a total of N pictures- one for each friend) are lying in two stacks in an arbitrary order. In one move, Sophie takes several (one or more) consecutive photos from the top of any stack and places them on top of the other stack without changing the order. Is Sophie always able to arrange the photos into one stack with alphabetical order of friends' names (listing from the bottom up) after no more than $2N + 1$ moves? (*S. Pavlov*)

Answer: yes; in fact, even $2N - 1$ moves are enough.

Solution. On the first move, we make one stack out of two (let's call it the *left* one).
On the second move, we relocate Amelia's photo and all photos above it to the right stack. After 2 moves, Amelia's photo ended up at the bottom of the right stack.
On the third move, we relocate everything from the right stack, except Amelia's photo, to the left stack.
On the fourth move, we relocate Bianca's photo and all photos above it to the right stack. As a result, after the 4th move, Amelia's and Bianca's photos are in place (at the bottom of the right stack).
We continue to act in this way, spending two moves for each subsequent photo. (Perhaps, in some cases, we won't have to move anything, then we'll save moves.) After no more than $2(N - 1)$ moves, it turns out that the bottom of the right stack contains $N - 1$ photos in the right order. If Yana's photo is in the left stack, then the last $(2N - 1)$ move will transfer it to the right one, and if it is in the right stack, then it is already in the right place (at the top).

Criteria. Only the answer – 0 points. Any reasoning leading to the answer "no" is worth 0 points.
The idea of putting the first photo in place in two moves – 3 points. Forgot to put Yana's photo back in place – a penalty of 1 point.

6. Several children came to participate in the final stage of an Olympiad.
   - Among them, there are two girls, each of whom knows exactly four boys.
   - There are three girls, each of whom knows exactly three boys.
   - Each of the remaining girls (if any) knows exactly two boys.
   - No two children of the same gender know each other.
   - If two children know each other, each of them can give a cheat sheet to the other one.

   It turned out that any girl could send a cheat sheet to any boy (even one she doesn't know, through other children), but if the organizers start closely monitoring at least one pair of children who know each other, this possibility would break down (meaning there would be some boy and girl who would no longer be able to send a cheat sheet to each other). Who came to the Olympiad in a greater number, boys or girls, and by how many? (*L. Koreshkova, P. Mulenko*)

Answer: there are 8 more boys than girls.

Solution. Let us denote the number of girls with two acquaintances as x, then the total number of acquaintances is $2 \cdot 4 + 3 \cdot 3 + x \cdot 2 = 17 + 2x$. On the other hand, if we draw a graph describing acquaintances (where children are vertices and acquaintances are edges), then this graph will turn out to be a tree. Indeed, the graph was initially connected, since each girl could pass the cheat sheet to any boy (and, via him, to any other girl), but deleting any edge makes it disconnected, which is one of the definitions of a tree. Therefore, the number of edges (i. e. acquaintances) is 1 less than the number of vertices: $17 + 2x = 2 + 3 + x + m - 1$, where m is the number of boys who came to the Olympiad. Thus, $m = x + 13 = (x + 5) + 8$, where $x + 5$ is the total number of girls who came to the Olympiad, that is, 8 more boys came.

Criteria. The remark that there is 1 more children than acquaintances is worth 3 points. If calculations show that there are 13 more boys, due to forgotten girls with 3 and 4 acquaintances, we give 5 points. 2 points if the number of acquaintances is expressed through the number of girls.
A solution of the form "let's draw 5 girls and 13 boys, and then add +1D, +1M" – 2 points.
Only the correct answer with an example – 1 point.

7. Alex, Walter, Ruby, and Roxanne are forming natural 5-digit numbers consisting of distinct non-zero digits.
   - Alex writes down all the numbers where the first digit is 1.
   - Walter writes down all the numbers where the first two digits are 1 and 2, in any order.
   - Ruby writes down all the numbers where the first three digits are 1, 2, and 3, in any order.
   - Roxanne writes down all the numbers where the first four digits are 1, 2, 3, and 4, in any order.

   How many five-digit numbers composed of distinct non-zero digits did not appear in any of their lists? (*L. Koreshkova*)

Answer: 13075 numbers.

Solution. There are $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ five-digit numbers with different digits.
First, Alex will write down all the numbers that start with 1: $1 \cdot (8 \cdot 7 \cdot 6 \cdot 5) = 1680$ numbers.
Then Walter will additionally write down the numbers that start with 21: $1 \cdot 1 \cdot (7 \cdot 6 \cdot 5) = 210$ numbers.
After that, Ruby will add the numbers that start with 231, 312, and 321, for a total of $3 \cdot (6 \cdot 5) = 90$ numbers.
Finally, Roxanne will add the numbers that start with 2341, 2413, 2431, 3412, 3421, 3142, 3241, as well as the numbers whose first digit is 4, followed by the digits 1, 2 and 3 in any order: $(7 + 1 \cdot 3 \cdot 2 \cdot 1) \cdot 5 = 65$ numbers.
In total, $15120 - 1680 - 210 - 90 - 65 = 13075$ numbers will not be written out.

Criteria. Calculation of one of the quantities: all numbers, Alex's, Walter's, Ruby's, and Roxanne's numbers – one point each (but no more than 3 in total). A solution where everything is calculated correctly but repetitions are not taken into account – 3 points. Arithmetic errors – penalty of 1 point for each one. Incorrect consideration of intersections or permutations – penalty of 2 points.

International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2024/2025. Qualifying round

Solutions for grade R9

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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1. Several kids decided to hold a series of elimination tests, the winner of which would earn the title of the class king. In an interview with the school newspaper, the winner said the following:
   (a) In the first and last tests combined, the same number of participants were eliminated as in all the others combined.
   (b) The number of eliminated participants in the second test was the same as in all subsequent tests in total.
   (c) In the first test, the least number of participants were eliminated.
   Prove that he made a mistake somewhere. (*P. Mulenko*)

   Remark. There was the following addition to the condition for this problem: there were at least four tests; the number of those who dropped out in the first test is **strictly** less than in any other.

   Solution. Let's denote the number of those who dropped out in the first test by a, in the second by b, in the third by c, in the last by z, in the rest (if there were any) by M. The first two conditions are written as follows: $a + z = b + c + M$, $b = c + M + z$. Substituting the second into the first, we get that $a + z = 2c + 2M + z$, that is, $a = 2(c + M)$. But this contradicts the last condition, according to which a must be less than c.

   Remark. The addition that there are at least four tests is essential. Otherwise, for example, the following case would be suitable: there are only 3 tests, 0 people dropped out in the first one, and the same number dropped out in the third as in the second. As an exercise, try to figure out what exactly is incorrect in the above solution if there are only three tests.

   Criteria. 1 point if the condition is rewritten in terms of inequalities.

2. A construction set consists of white cubes. Paul assembles a large cube from all the cubes, then selects 4 faces of the large cube and paints them red. After that, he disassembles the large cube and counts the cubes that have at least one face painted red. Paul ends up with more than 500 but less than 600 such cubes. How many exactly? Find all possibilities. (*L. Koreshkova*)

   Answer: 518 or 528 cubes.

   Solution. Let an edge of the large cube consist of n edges of small cubes. There are two ways to paint the 4 faces of a cube: a “ring”, leaving two opposite faces unpainted, and a “shelf”, leaving two adjacent faces unpainted.

   In a “ring”, the total number of painted cubes is equal to the number of cubes in the 4 faces minus the number of cubes in the 4 edges, which were counted twice, that is $4n^2 - 4n$.

   In a “shelf”, the total number of painted cubes is equal to the number of cubes in the 4 faces minus the number of cubes in the 5 common edges plus 2 vertices of the cube, where the triplets of faces and edges meet (since we have previously counted them 3 times in the faces and subtracted them 3 times in the edges), that is $4n^2 - 5n + 2$.

   Then we can act in different ways.

   The first method. If $n \leq 11$, then Paul painted less than $4n^2 \leq 484$ cubes, and if $n \geq 13$, then more than $4n^2 - 5n \geq 4 \cdot 13^2 - 5 \cdot 13 = 169 \cdot 4 - 65 > 600$ (the greater n, the more cubes are painted in any way).

   Thus, from 500 to 600 cubes could only be obtained with $n = 12$. In this case, for the “ring” coloring, we get $4 \cdot (12^2 - 12) = 528$ painted cubes, and for the “shelf” coloring, we get $4 \cdot 12^2 - 5 \cdot 12 + 2 = 518$ cubes.

The second method. We need to find solutions in natural numbers for at least one of the double inequalities on the number of colored cubes:

$$\left[\begin{aligned} 500 \le \quad 4(n^2-n) &\le 600,\\ 500 \le \quad 4n^2-5n+2 &\le 600. \end{aligned}\right. \Leftrightarrow \left[\begin{aligned} &\begin{cases} 4n^2-4n-500 \ge 0,\\ 4n^2-4n-600 \le 0, \end{cases}\\ &\begin{cases} 4n^2-5n-498 \ge 0,\\ 4n^2-5n-598 \le 0. \end{cases} \end{aligned}\right.$$

Direct solution of both systems of inequalities gives only one natural solution (namely $n = 12$), which fits to both systems.

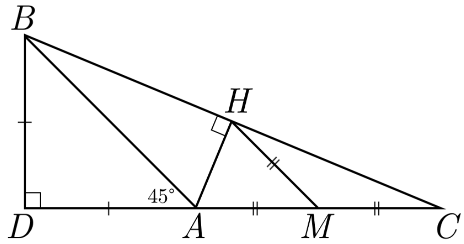
Criteria. 3 points if it is proved that $n = 12$. Two more points for each answer. The system is written correctly, but not solved – 2 points. Solving the problem in only one of the cases (ring/shelf) – 4 points.

3. In a triangle ABC, $\angle A = 135^\circ$, $AB = 14$, $BC = 26$. Point H is the foot of the altitude from point A, and M is the midpoint of AC. Find HM. (*L. Koreshkova*)

Answer: $5\sqrt{2}$.

Solution. Extend CA beyond A and drop perpendicular BD onto the resulting ray.

$\angle BAD = 180^\circ - 135^\circ = 45^\circ$, that is, $\triangle BAD$ is an isosceles right triangle, and $BD = DA = 7\sqrt{2}$. Triangle BCD is also a right triangle, so $CD = \sqrt{BC^2 - BD^2} = 17\sqrt{2}$, whence $AC = DC - DA = 10\sqrt{2}$. HM is the median from the vertex of the right angle of right triangle AHC, so $HM = AC/2 = 5\sqrt{2}$.



4. Solve the equation $[x]^2 + \{x\}^2 = 2x^2$. (Here $[x]$ and $\{x\}$ are the integer and fractional parts of x.) (*S. Pavlov*)

Answer: $0,\ 1-\sqrt{3},\ 2-2\sqrt{3},\ 3-3\sqrt{3}$.

Solution. Let's rewrite the condition using that $x = [x] + \{x\}$, and we get: $[x]^2 + \{x\}^2 = 2[x]^2 + 2\{x\}^2 + 4[x]\{x\}$, that is, $[x]^2 + \{x\}^2 + 4[x]\{x\} = 0$. We see a trivial solution $x = 0$; all other roots must have negative integer part (otherwise the left side is positive).
We divide the equation by $[x]^2$ (because $[x] \ne 0$), make the substitution $t = \frac{\{x\}}{[x]}$ and get a quadratic equation $t^2 + 4t + 1 = 0$ with roots $t_{1,2} = -2 \pm \sqrt{3}$.
The root $t_1 = -2 - \sqrt{3}$ is not suitable, since then $\{x\} \ge 2 + \sqrt{3} \ge 1$. After the reverse substitution $\frac{\{x\}}{[x]} = -2 + \sqrt{3}$, with and the condition $0 \le \{x\} < 1$, it turns out that the possible values for the integer part are $[x] \in \{-1; -2; -3\}$, whence $x \in \{0,\ 1-\sqrt{3}, 2-2\sqrt{3}, 3-3\sqrt{3}\}$.

Criteria. It is proved that $x \ge 0$ – 2 points. The quadratic equation is solved correctly – 2 points. Each missing root – a penalty of 1 point.

5. A fraction of the form $\dfrac{1}{n^2}$, where n is a natural number, is called *uni-square*. Find the maximum uni-square fraction that can be expressed as the sum of two uni-square fractions. (*S. Pavlov, A. Tesler*)

Answer: $\dfrac{1}{144}$ (can be expressed as $\dfrac{1}{12^2} = \dfrac{1}{15^2} + \dfrac{1}{20^2}$).

Solution. Let $\frac{1}{n^2} = \frac{1}{p^2} + \frac{1}{q^2}$, then $p^2q^2 = p^2n^2 + q^2n^2$. In other words, (pn, qn, pq) is a Pythagorean triple. Let g denote the greatest common divisor of pn and qn. Then g divides pq, $(a, b, c) = (\dfrac{pn}{g}, \dfrac{qn}{g}, \dfrac{pq}{g})$ is a primitive Pythagorean triple (of pairwise coprime numbers), $n = \sqrt{\dfrac{gab}{c}}$, $p = \sqrt{\dfrac{gac}{b}}$, $q = \sqrt{\dfrac{gbc}{a}}$.
Since the radicands must be integers and a, b, c are pairwise coprime, then all three numbers a, b, c divide g.
Since a or b is divisible by 3 (otherwise c^2 would leave a remainder of $a^2 + b^2 \equiv 2$ modulo 3, which is impossible), then n is also divisible by 3. Moreover, a or b is divisible by 4 (otherwise $a^2 + b^2$ would leave a remainder of 1 or 5 modulo 8, but these are not remainders of perfect squares), so n is divisible

by 4 (the expression under the radical is divisible by 4 thanks to a or b, and by another 4 thanks to g). Therefore, $n \geq 12$, and the case $n = 12$ is realized for $a = 3$, $b = 4$, $c = 5$, $g = 60$, $p = 15$, $q = 20$.

Criteria. 3 points for the example, 4 points for an estimation.

6. Several children came to participate in the final stage of an Olympiad.
   - Among them, there are two girls, each of whom knows exactly four boys.
   - There are three girls, each of whom knows exactly three boys.
   - Each of the remaining girls (if any) knows exactly two boys.
   - No two children of the same gender know each other.
   - If two children know each other, each of them can give a cheat sheet to the other one.

   It turned out that any girl could send a cheat sheet to any boy (even one she doesn't know, through other children), but if the organizers start closely monitoring at least one pair of children who know each other, this possibility would break down (meaning there would be some boy and girl who would no longer be able to send a cheat sheet to each other). Who came to the Olympiad in a greater number, boys or girls, and by how many? (*L. Koreshkova, P. Mulenko*)

   Answer: there are 8 more boys than girls.

   Solution. Let us denote the number of girls with two acquaintances as x, then the total number of acquaintances is $2 \cdot 4 + 3 \cdot 3 + x \cdot 2 = 17 + 2x$. On the other hand, if we draw a graph describing acquaintances (where children are vertices and acquaintances are edges), then this graph will turn out to be a tree. Indeed, the graph was initially connected, since each girl could pass the cheat sheet to any boy (and, via him, to any other girl), but deleting any edge makes it disconnected, which is one of the definitions of a tree. Therefore, the number of edges (i. e. acquaintances) is 1 less than the number of vertices: $17 + 2x = 2 + 3 + x + m - 1$, where m is the number of boys who came to the Olympiad. Thus, $m = x + 13 = (x + 5) + 8$, where $x + 5$ is the total number of girls who came to the Olympiad, that is, 8 more boys came.

   Criteria. The remark that there is 1 more children than acquaintances is worth 3 points. If calculations show that there are 13 more boys, due to forgotten girls with 3 and 4 acquaintances, we give 5 points. 2 points if the number of acquaintances is expressed through the number of girls.

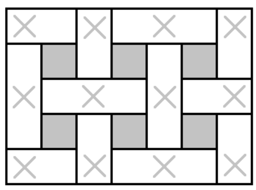
   A solution of the form "let's draw 5 girls and 13 boys, and then add +1D, +1M" – 2 points.

   Only the correct answer with an example – 1 point.

7. Is it possible to cut a 100×100 square along the cell boundaries into 2024 rectangles so that the union of any set of 2 to 2023 rectangles is not a rectangle? (*A. Tesler*)

   Answer: yes.

   Solution. Let's present one of the possible constructions. Consider a rectangle $(2n + 1) \times (2m + 1)$ with odd sides, where $nm \geq 2$. Let's split it into rectangles 1×1, 1×2, 1×3, 2×1, 3×1 as in the figure (where $n = 2$ and $m = 3$). That is, let's cut out squares at the intersection of rows and columns with even numbers (shown in gray). Then let's place rectangles 1×3 and 3×1 between adjacent such squares in an alternating manner (rectangles of two cells are obtained by cutting off the boundaries of the large rectangle). Note that the resulting pieces (except for the squares) are in one-to-one correspondence with the cells marked with a cross. Therefore, there are $nm + (n + 1)(m + 1)$ pieces.



   Moreover, under the constraint $nm \geq 2$, there is no rectangle consisting entirely of the resulting small rectangles, distinct from them and from the entire large rectangle. Indeed, if it contains a square, then it also contains some of its neighbors, and then it contains all its neighbors and the nearest squares. If it contains some rectangle, then it is easy to check that it also contains the square adjacent to it.

   For $\{n, m\} = \{28, 35\}$, we get 2024 rectangles, and $2n + 1$ and $2m + 1$ are no more than 100. Now we can expand the first row and the first column to obtain the required cutting of the square 100×100.

   Criteria. If it is not proven that the given example satisfies the conditions, then no more than 5 points are given.

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"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Qualifying round

Solutions for grade R10

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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1. In classes 10A and 10B, there are 30 students each. The average height of the boys in class 10A is greater than that of the boys in class 10B. The average height of the girls in class 10A is greater than that of the girls in class 10B. Is it possible that the average height of all students in class 10A is less than that of all students in class 10B? (*A. Tesler*)

   Solution. Yes. For example, let's say that in class 10A there are 10 boys and 20 girls, and in 10B, vice versa. Also let's say that all the boys from 10A are 181 cm tall, the boys from 10B are 180 cm tall, the girls from 10A are 175 cm tall, and the girls from 10B are 174 cm tall. Then the average height in 10A is 177 cm, and in 10B it is 178 cm.

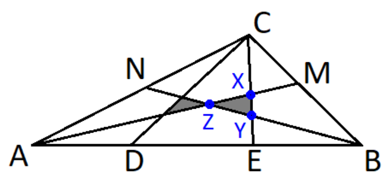
2. The function f is defined by the formula $f(x) = \dfrac{2x}{3x^2+1}$. Prove that for any two real mutually inverse numbers s and t, the sum $f(s) + f(t)$ does not exceed 1. (*S. Pavlov*)

   Solution. Let $s \neq 0$, $t = s^{-1}$ and $u = s + s^{-1}$, then

$$f(s)+f(t) = \frac{2s}{3s^2+1} + \frac{2s^{-1}}{3s^{-2}+1} = \frac{8s+8s^{-1}}{(3s^2+1)(3s^{-2}+1)} = \frac{8u}{3(s^2+s^{-2})+10} = \frac{8u}{3(u^2-2)+10} = \frac{8u}{3u^2+4}.$$

   It is required to prove that $f(s) + f(t) \le 1$, that is, $3u^2 - 8u + 4 \ge 0$. This quadratic inequality is false only for $\frac{2}{3} < u < 2$. But u is either negative (for $s < 0$) or not less than 2 (for $s > 0$, this follows from the inequality of means). critThe idea of the substitution is worth 2 points. If it is not taken into account that s and t can be negative, no more than 5 points are given.

3. In a triangle ABC, two medians AM and BN are drawn. The third vertex C is connected to points D and E, which divide AB into three equal parts. What fraction of the area of the triangle ABC do the two shaded triangles occupy? (*L. Koreshkova*)



   Answer: $\dfrac{1}{30}$.

   Solution. Let's denote $X = AM \cap CE$, $Y = BN \cap CE$, $Z = AM \cap BN$. From Menelaus' theorem, applied to $\triangle AMB$ and line CE, it follows: $AX : XM = 4 : 1$. From the same theorem, applied to $\triangle ANB$ and line CE, it follows that $NY = YB$. Moreover, $AZ : ZM = ZB : ZN = 2 : 1$ by the property of medians. Since $S_{ZMB} = \frac{1}{6}S_{ABC}$ (the medians divide any triangle into six equal parts), then

$$S_{XYZ} = \frac{1}{2}ZX \cdot ZY \cdot \sin\angle MZB = \frac{1}{2} \cdot \frac{2}{5}ZM \cdot \frac{1}{4}ZB \cdot \sin\angle MZB = \frac{1}{10}S_{ZMB} = \frac{1}{60}S_{ABC}.$$

   The area of the second shaded triangle is equal to the area of the first and can be found in the same way.

   Criteria. For the formulation of Menelaus' theorem or for the statement about the preservation of the ratio of areas under affine transformations, 2 points are awarded (they are not added up).

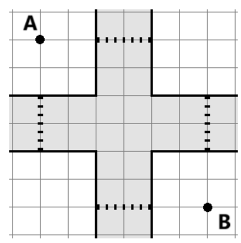
4. In a bag for the game Bingo, there are 10 counters with the following numbers: 1, 2, 3, 5, 7, 10, 20, 30, 53, 75. Three counters are taken out of the bag, and the largest number that can be formed by arranging them is recorded. For example, if you take out the counters 7, 20, 30, you record the number 73020. How many numbers greater than 2024 can be recorded? (*L. Koreshkova*)

   Answer: $C_{10}^3 - C_5^3 - 6 = 104$.

Solution. Let's sort the counters as follows: 7, 75, 5, 53, 3, 30, 2, 20, 1, 10. Note that in the written number the counters are in the same order in which they are sorted (i.e. if you rearrange any two counters, the number composed of them will decrease). If you take 3 single-digit counters, the number composed of them will be less than 2024, and in other cases it is strictly greater: if it is five- or six-digit, then this is obvious; if the first digit is greater than 2, this is also obvious, and only two cases remain: $2|20|1$ and $2|1|10$. In total, $C_{10}^3 - C_5^3 = 110$ ways to choose counters.
However, some numbers correspond to more than one set of counters (for example, 7553 is obtained from the sets 75, 5, 3 and 7, 5, 53). It remains to find all such numbers. Let us split a number into two parts consisting of counters 7 through 3 and 30 through 10. Note that the second part is allocated and split into counters in a unique way: counters 10, 20, and 30 are restored using zeros, after which 1 and 2 are also allocated uniquely.
The first part is a number consisting of the digits 7, 5, 3, which are non-strictly decreasing. Possible ambiguities when splitting into counters arise only because $75 = 7|5$ and $53 = 5|3$. Since we need partitions of the same length, the only options are $75|3 = 7|53$ (with one of the 5 counters added to the second part) and $75|5|3 = 7|5|53$ (with the second part empty). So we should subtract 6 from the answer.

Criteria. 1 point for indicating which sets of counters are suitable, and another 2 points are added for calculating the amount of ways to choose counters. All options found with ambiguous decomposition into counters are worth 2 points (1 point if at least one of them is found).

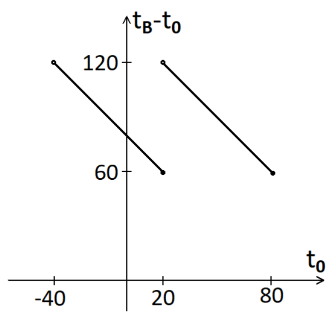
5. The plan shows an intersection of Horizontal and Vertical streets (the side of each cell is 5 meters, and the crossings are shown as dotted lines). The traffic lights alternate with a period of 2 minutes according to the following schedule:
   - 40 seconds – green light for pedestrians crossing the Horizontal street;
   - the next 20 seconds – red for all pedestrians;
   - the following 40 seconds – green for those crossing the Vertical street;
   - and the last 20 seconds – red for all pedestrians.

   Edgar walks at a speed of 1 m/s. At a random moment, he finds himself at point A, from where he crosses to point B in the fastest way possible, without breaking the rules. Edgar can see how much time is left until the change of signal at each traffic light, so he does not start crossing the street if he cannot finish in time. How many seconds on average will it take Edgar to reach point B? (*A. Tesler*)

Answer: 90 seconds.

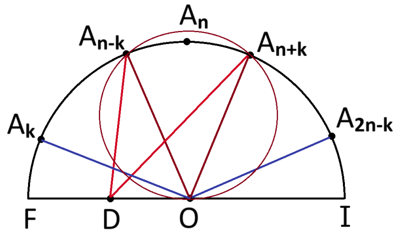
Solution. Let the traffic light across Horizontal Street turn green at time $t = 0$ (in seconds). Then if the initial time is $-40 < t_0 \le 20$, then Edgar manages to cross Horizontal Street by time $t_1 \in [10; 40)$, starts crossing Vertical Street at $t = 60$ and arrives at point B at time $t_B = 80$. It turns out that he spent from 120 to 60 seconds, and since the dependence is linear (see graph), the average time is 90 seconds. At $20 < t_0 \le 80$ everything is the same, but the streets are crossed in a different order, and the average in this interval is also 90 seconds. Then everything is repeated, since the period of alternating is 120 seconds.



6. Segment FI is the diameter of a semicircle which is divided into equal arcs by points $A_1, A_2, \ldots, A_{2n-1}$ (where $n > 2$). A point D is marked on the segment FI, and it turns out that for some k $(1 \le k < \frac{n}{2})$, the sines of the angles A_kDA_{2n-k} and $A_{n-k}DA_{n+k}$ are equal. Prove that D is the midpoint of FI. (*P. Mulenko*)

Solution. The angles obviously cannot be equal ($k < n - k$, so the second angle lies inside the first). Therefore, the equality of sines means that the angles add up to 180°. Let us denote the midpoint of the segment FI by O and note that the angle A_kOA_{2n-k}, as the central one, is equal to $2n - 2k$ arcs, and the angle $A_{n-k}OA_{n+k}$ is equal to $2k$ arcs, that is, together they will give the entire semicircle, and, thus, will be equal to 180°.

Now we will show that if the point D does not coincide with O, then the sum will be less than 180° (and, therefore, their sines will not be equal). Let's construct the circumscribed circles of triangles A_kOA_{2n-k} and $A_{n-k}OA_{n+k}$ (the second of them is shown in the figure). Since the ray $OA_n \perp FI$, the centers of both circles will lie on the segment OA_n, and the segment FI will touch them (moreover, the central angles of the large circles will become inscribed for the small ones).



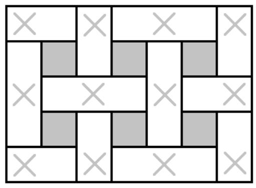
But the point D is guaranteed to be outside the constructed circles, and each of the angles under consideration will be smaller than the corresponding inscribed angle with the vertex at O.

Criteria. For mentioning that the angles cannot be equal, 2 points are given.

7. Is it possible to cut a 100×100 square along the cell boundaries into 2024 rectangles so that the union of any set of 2 to 2023 rectangles is not a rectangle? (*A. Tesler*)

Answer: yes.

Solution. Let's present one of the possible constructions. Consider a rectangle $(2n+1) \times (2m+1)$ with odd sides, where $nm \geq 2$. Let's split it into rectangles 1×1, 1×2, 1×3, 2×1, 3×1 as in the figure (where $n = 2$ and $m = 3$). That is, let's cut out squares at the intersection of rows and columns with even numbers (shown in gray). Then let's place rectangles 1×3 and 3×1 between adjacent such squares in an alternating manner (rectangles of two cells are obtained by cutting off the boundaries of the large rectangle). Note that the resulting pieces (except for the squares) are in one-to-one correspondence with the cells marked with a cross. Therefore, there are $nm + (n+1)(m+1)$ pieces.



Moreover, under the constraint $nm \geq 2$, there is no rectangle consisting entirely of the resulting small rectangles, distinct from them and from the entire large rectangle. Indeed, if it contains a square, then it also contains some of its neighbors, and then it contains all its neighbors and the nearest squares. If it contains some rectangle, then it is easy to check that it also contains the square adjacent to it.

For $\{n, m\} = \{28, 35\}$, we get 2024 rectangles, and $2n+1$ and $2m+1$ are no more than 100. Now we can expand the first row and the first column to obtain the required cutting of the square 100×100.

Criteria. If it is not proven that the given example satisfies the conditions, then no more than 5 points are given.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Qualifying round

Solutions for grade R11

Each problem is scored at 7 points. A score of 1–3 points means that the problem is generally unsolved, but there are significant advances; 4–6 points mean that the problem is generally solved, but there are significant shortcomings.

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1. Solve the equation $[x]^2 + 2\{x\}^2 = 2x^2$. (Here $[x]$ and $\{x\}$ are the integer and fractional parts of x.) (*S. Pavlov*)

Answer: $x \in \{-2{,}25, -1{,}5, -0{,}75\} \cup [0, 1)$.

Solution. Let's introduce notations $y = [x]$ and $z = \{x\}$, then y is an integer, $0 \le z < 1$, and from the equation we get $y(y+4z) = 0$. If $y = 0$, then any z from the half-interval is suitable. If $0 \ne y = -4z$, then $y \in \{-3, -2, -1\}$ and in each case we get a unique solution.

Criteria. If the participant believes that $[x]$ is the result of rounding to the nearest integer (not to the smaller one) and $\{x\} = x - [x]$, then 2 points are awarded for an otherwise complete solution (with the answer $x \in (-1, 1)$).

2. Connor and Mary take turns placing the numbers $+1$ and -1 in the squares of an 8×8 board (Connor starts) according to the following rules.
   - Before each move, the player chooses a number and a color (red, green, or blue) and writes the chosen number in any free square on the board.
   - If, after a player's move, the sum of the numbers of the chosen color equals zero, that player automatically loses.
   - After the board is filled, the sums of the numbers of each color are calculated. If the product of these sums is positive, Connor wins; if negative, Mary wins; if any color was never used, it is not included in the count.

   Who can ensure their victory, and how should they act to win? (*P. Mulenko*)

Answer: Mary can ensure her victory.

Strategy. Let's say Connor's first move was $+1$ in blue. This means that the sum of the blue numbers will always be positive (to change this, one of the players would have to make it equal to zero first, which is automatically considered a loss). So Mary has no choice but to repeat Connor's moves in the same color until Connor decides to place a number *in the second color*, after which she will have to place a number *in the third color* so that the sum in one or three colors ends up being negative (which will ensure a negative product). After that, the course of the game does not matter, Mary only needs to make sure that her moves do not make the sum in any of the colors equal to zero.

The only exception is if Connor never places a number in the second color. In this situation, Mary will have to put the number in the second color as her last move so that the sum is negative in exactly one of the two colors.

If Connor put -1 as his first move, Mary does the same: when adding the second color, she adds the third with the right sign, otherwise, she puts $+1$ in a different color as her last move.

Criteria. For noticing that the sum of numbers of the same color is determined by the first addend, 2 points are given. For the strategy in an exceptional case, 1 point is given. If the strategy is described, but the exception is forgotten, then no more than 5 points are given for the problem.

3. A fraction of the form $\frac{1}{n^2}$, where n is a natural number, is called *uni-square*. Find the maximum uni-square fraction that can be expressed as the sum of two uni-square fractions. (*S. Pavlov, A. Tesler*)

Answer: $\frac{1}{144}$ (can be expressed as $\frac{1}{12^2} = \frac{1}{15^2} + \frac{1}{20^2}$).

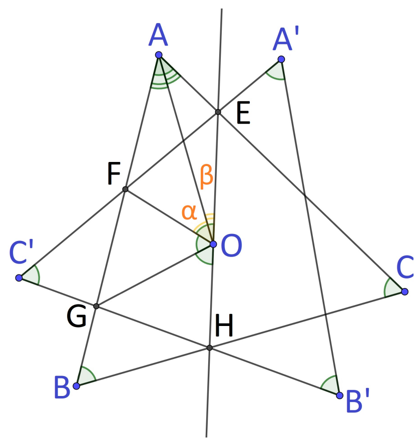
Solution. Let $\frac{1}{n^2} = \frac{1}{p^2} + \frac{1}{q^2}$, then $p^2q^2 = p^2n^2 + q^2n^2$. In other words, (pn, qn, pq) is a Pythagorean triple. Let g denote the greatest common divisor of pn and qn. Then g divides pq, $(a, b, c) = (\frac{pn}{g}, \frac{qn}{g}, \frac{pq}{g})$ is a primitive Pythagorean triple (of pairwise coprime numbers), $n = \sqrt{\frac{gab}{c}}$, $p = \sqrt{\frac{gac}{b}}$, $q = \sqrt{\frac{gbc}{a}}$. Since the radicands must be integers and a, b, c are pairwise coprime, then all three numbers a, b, c divide g.
Since a or b is divisible by 3 (otherwise c^2 would leave a remainder of $a^2 + b^2 \equiv 2$ modulo 3, which is impossible), then n is also divisible by 3. Moreover, a or b is divisible by 4 (otherwise $a^2 + b^2$ would leave a remainder of 1 or 5 modulo 8, but these are not remainders of perfect squares), so n is divisible by 4 (the expression under the radical is divisible by 4 thanks to a or b, and by another 4 thanks to g). Therefore, $n \geq 12$, and the case $n = 12$ is realized for $a = 3$, $b = 4$, $c = 5$, $g = 60$, $p = 15$, $q = 20$.

Criteria. 3 points for the example, 4 points for an estimation.

4. A line that does not pass through the vertex is drawn through the center of an equilateral triangle with an area of 1. If the triangle is folded along this line, a certain quadrilateral is covered twice. What is the minimum possible area of this quadrilateral? (*L. Koreshkova*)

Answer: 1/3.

Solution. Let us denote the vertices of the triangle by A, B, C so that the line separates AB from C, the points symmetrical to them are A', B', C'. The intersection points of AC with $A'C'$, AB with $A'C'$, AB with $B'C'$, BC with $B'C'$ we denote by E, F, G, H respectively (see the figure).
We need to find the minimum area of $EFGH$. Note that the triangles OEF, OFG, OGH are equal: OEF and OGH are obtained from OFG by reflection across to EH and rotation by 120° in different directions. Therefore, it is sufficient to find the minimum area of OEF. Let α and β denote the angles FOA and EOA and note that the area OEF is proportional to $|OF| \cdot |OE|$.
Since $|OA|$ and the angles OAF, OAE are constant, then by the sine law $|OF|$ is proportional to



$$\frac{1}{\sin\left(\pi - \frac{\pi}{6} - \alpha\right)} = \frac{1}{\cos(\beta)},$$

similarly $|OE|$ is proportional to $1/\cos(\alpha)$. Since $\alpha + \beta = \pi/3$, it is convenient to introduce the variable $\gamma = \alpha - \pi/6$.
Thus, $\alpha = \gamma + \pi/6$, $\beta = \gamma - \pi/6$, γ can take values from $-\pi/6$ to $\pi/6$ inclusive, and we are interested in the maximum of the expression

$$\cos\left(\gamma - \frac{\pi}{6}\right) \cdot \cos\left(\gamma + \frac{\pi}{6}\right) = \cos^2\gamma - \frac{1}{4}.$$

It is achieved at $\gamma = 0$ (that is, $OE \parallel AB$), which gives the minimum area of 1/3.

Criteria. 2 points for the answer with the example, 5 points for an estimation.

5. The plan shows an intersection of Horizontal and Vertical streets (the side of each cell is 5 meters, and the crossings are shown as dotted lines). The traffic lights alternate with a period of 2 minutes according to the following schedule:
   - 40 seconds – green light for pedestrians crossing the Horizontal street;
   - the next 20 seconds – red for all pedestrians;
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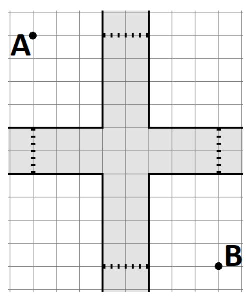
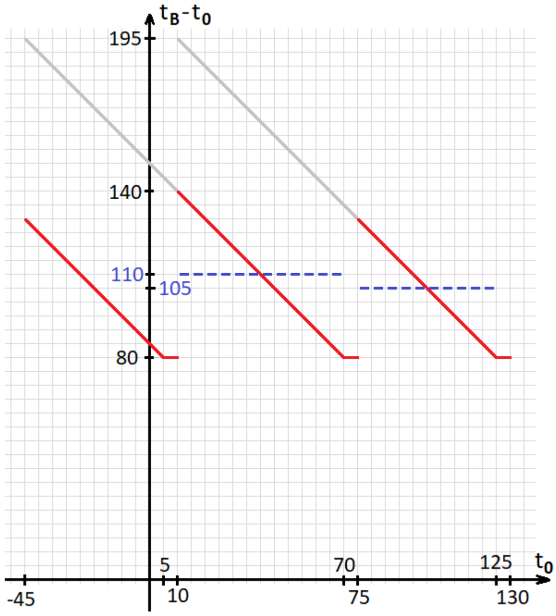


Edgar walks at a speed of 1 m/s. At a random moment, he finds himself at point A, from where he crosses to point B in the fastest way possible, without breaking the rules. Edgar can see how much time is left until the change of signal at each traffic light, so he does not start crossing the street if he cannot finish in time. How many seconds on average will it take Edgar to reach point B? (*A. Tesler*)

Solution. Let the traffic light across Horizontal Street turn green at time $t = 0$ (in seconds). Let's calculate how many seconds Edgar will spend depending on the exit time t_0. Note that before the first and after the second crossing, it will be fastest for Edgar to walk along the segments parallel to the streets, and between the two crossings, along the hypotenuse, which will take him $\sqrt{15^2 + 20^2} = 25$ seconds. Approaching the road, Edgar needs to wait for the right time to cross ($120n \le t \le 120n + 30$ for Horizontal Street and $120n + 60 \le t \le 120n + 90$ for Vertical Street).

The following cases are possible (including those where Edgar crosses the roads in a non-optimal order).

- Edgar first crosses Vertical Street, and he does not have to wait for the traffic light neither at the first crossing ($120n + 45 \le t_0 \le 120n + 75$), nor at the second one ($120m - 50 \le t_0 \le 120m - 20$). Then he needs 80 seconds, and t_0 lies in the interval $120n + 70 \le t_0 \le 120n + 75$.
- Edgar first crosses Vertical Street and he doesn't have to wait for the first traffic light, but he waits for the second one. Then he needs from 105 to 80 seconds. In this case, t_0 lies in the interval $120n + 45 \le t_0 < 120n + 70$ and the travel time depends on t_0 linearly in each period.
- Edgar first crosses the Vertical Street, but he has to wait for both traffic lights. He will spend from 195 to 105 seconds in linear dependence on $t_0 \in (120n - 45; 120n + 45)$.



- If Edgar first crosses the Horizontal Street and does not have to wait for traffic lights, then he spends 80 seconds and $120n + 5 \le t_0 \le 120n + 10$.
- Edgar first crosses the Horizontal Street and waits only at the second traffic light. Then he needs from 105 to 80 seconds in linear dependence on $t_0 \in [120n - 20; 120n + 5)$.
- Edgar first crosses the Horizontal Street and waits at both traffic lights, then he needs from 195 to 105 seconds in linear dependence on $t_0 \in (120n + 10; 120n + 100)$.

Let's find the average value in the interval $t_0 \in (120n + 5; 120n + 125)$. Since Edgar acts optimally, he needs 80 seconds at $120n + 5 \le t_0 \le 120n + 10$, 140 to 80 seconds at $120n + 10 \le t_0 \le 120n + 70$ (an average of 110 seconds, since the dependence is linear), 80 seconds at $120n + 70 \le t_0 \le 120n + 75$, and 130 to 80 seconds at $120n + 75 \le t_0 \le 120n + 125$ (an average of 105, since the dependence is linear).

So on average it will take $\dfrac{80 \cdot 5 + 110 \cdot 60 + 80 \cdot 5 + 105 \cdot 50}{120} = \dfrac{1265}{12} = 105\frac{5}{12}$ seconds.

Criteria. If the solution assumes that Edgar only moves horizontally and vertically, and the problem is solved correctly overall (with the answer $107\frac{1}{12}$ seconds), then 2 points are given.

6. In the school where Alice studies, marks 1, 2, 3, 4, and 5 are given. Alice received exactly 60 marks in the first quarter. By multiplying them, she obtained a number with digits sum 12. What is the maximum possible arithmetical mean of Alice's marks? (*A. Tesler*)

Answer: 4,65.

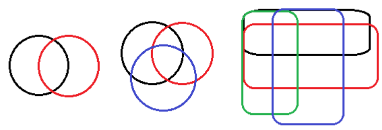
Solution. The product obtained by Alice has digits sum 12, which means it is divisible by 3, but not by 9. In this case, it has the form $3 \cdot 2^b \cdot 5^c$. Its decimal notation begins either with a number $3 \cdot 5^n$ (for $n = c - b \ge 0$), or with a number $3 \cdot 2^{-n}$ (for $n = c - b < 0$), and is continued by zeros. Let's consider the first case. For $n = 2$, the sum of digits is indeed equal to 12 (for $n = 0$ and $n = 1$, it is not). If $n > 2$, then 5^n ends in 25 (easy to prove by induction on n), and $3 \cdot 5^n$ ends in 75. But there are some other digits before 75, so the sum of digits is greater than 12. Thus, this case is realized only for $n = 2$. Then three of the marks are equal to 3, 5, 5. The product of the remaining 57 marks can be made equal to a power of ten: for example, if among them there are 38 fives and 19 fours, then it is equal to 10^{38}. In this case, the arithmetic mean is equal to $(3 + 19 \cdot 4 + 40 \cdot 5) : 60 = 4{,}65$. To further increase the mean, we need to increase the number of fives (since it is impossible to get rid of the three), but then the power of five (c) in the factorization will increase and the power of two (b) will decrease, that is, $n = c - b > 2$. For the same reason, with an average greater than 4,65, the second case (where $b > c$) is impossible.

Criteria. For indicating that the number has the form $3 \cdot 2^b \cdot 5^c$, 2 points are given. For the example, 3 points are given.

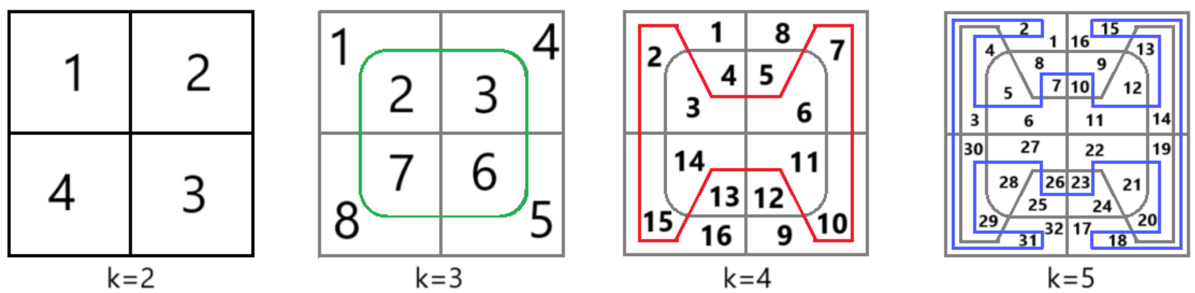
7. The Tester has a square Polygon. A Source situated in a known point of the Polygon emits 10 types of radiation, which spread along straight and curved paths but cannot cross barriers (each type of radiation has its own type of barrier). The radiation types are divided into good and bad (it is possible that all 10 are good or all 10 are bad), and the Tester does not know which types are good. The optimal life zone is the zone achievable by all types of good radiation but unreachable by the bad one.
The Tester is preparing for the experiment: he installs 10 barriers (one of each type) so that each barrier divides the Polygon into two parts. After that, he will turn on the Source, and the experiment will begin. The Tester wants the optimal life zone to be connected (i.e., not composed of several separate parts) and its area to be $\frac{1}{1024}$ of the area of the square. Can the Tester install the barriers in such a way to guarantee this? Two barriers can have only a finite number of common points, and a barrier cannot pass through the Source. (*A. Tesler*)

Answer: yes, it can.

Remark. A well-known mathematical object is the Euler–Venn diagrams for n sets with small n (see the figure on the right). It follows from the solution of this problem that such diagrams are possible for an arbitrary n.

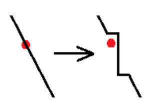


Solution. We will construct the barriers as shown in the figure below. The first two of them connect the midpoints of the opposite sides of the Polygon and divide it into 4 zones. Each subsequent barrier is a closed line that passes through each of the zones formed by the previous barriers and divides it into two equal parts. To draw such a line through all the zones, they must be numbered so that every two zones with adjacent numbers (as well as the last zone and the first one) have a common section of the border. We will construct such a numbering by induction (after k barriers have been drawn, the zones must be numbered from 1 to 2^k). For $k = 2$, we number the zones "in a circle" from 1 to 4. When adding a new barrier, we divide zone a into new zones $2a - 1$ and $2a$, and if a is odd, then zone $2a - 1$ will be outside the new barrier, and $2a$ inside; and if a is even, then vice versa. As a result, two new zones with neighboring numbers will either be halves of one old zone, or halves of neighboring old zones, not divided by the new barrier (i.e., still neighboring). Zone with number 2^k will always be outside the new barriers (as the *second* half of zone with *even* number 2^{k-1}), and also zone number 1 will (as the *first* half of zone with *odd* number 1), that is, they will remain neighbors.

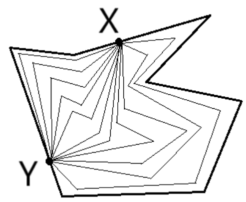


Of course, we will draw the barriers transversally (that is, at the intersection point, the first of the intersecting barriers goes to the other side of the second one: there is no touching or common fragments of non-zero length). It is also easy to avoid intersections of three barriers in one point.

If it turns out that one of the barriers passes through the Source, then we can move it, compensating for the change in area somewhere nearby (see the picture).



The number of zones doubles each time, so after 10 partitions we will get 1024 pieces. The optimal life zone corresponds to a set of 10 answers to the questions "Should we separate ourselves from the source by the barrier number i?". The answer to each question reduces the number of suitable pieces by half, i.e. after 10 questions there will be exactly one piece left. So the optimal life zone is connected.



It remains to understand why any zone can be divided into two equal parts (then after 10 divisions the area of each zone will be exactly equal to $\frac{1}{1024}$ of the total). The reason is demonstrated in the figure on the right: as the boundary shifts from left to right, the area of the left part continuously increases, so there is a moment when it will be equal to half the area of the entire figure.

In fact, many of the geometric considerations we used are not entirely rigorous, although they are intuitively obvious. For example, it is not very clear what is meant by "the area of the left side is continuously increasing" (in school, only continuous functions of one variable are studied – what function are we talking about here?). And in general, it is not clear what the area of an arbitrary figure is. Therefore, we will draw barriers only in the form of broken lines (we assume that the concept of the area of a polygon is known). Below, we prove the key geometric statement used in the solution.

Theorem. For any polygon P and two different points X, Y on its boundary, we can draw a simple broken line L with ends at X and Y, all other points of which lie strictly inside P, the areas of the two resulting polygons are equal, and L does not pass through a given point Z inside P.

Proof: induction on the number of vertices of P, which also include the points X and Y themselves (if they lie on the sides). In the case where P is a triangle, and X and Y are its vertices, L can be taken to be a two-link broken line with an intermediate vertex in the middle of the ceviana from the third vertex of P (the ceviana is chosen so that the resulting broken line does not pass through Z).

In the general case, it is claimed that there exist (distinct) vertices A and B of the polygon P such that all interior points of the segment AB lie strictly inside P, and X and Y lie on different arcs of the boundary of P (or coincide with A or B). If this can be proved, then we can cut the polygon by this segment (each of them will have strictly fewer vertices) and apply the induction hypothesis to the pairs of points (X, C) and (C, Y) for a suitable $C \in AB$ (different from Z).

Let's look at the neighboring vertices E, F with the vertex X. If there are no other vertices of the polygon in the triangle XEF (and $X \notin EF$), then we can take $A = E$, $B = F$. If there are other vertices there (and still $X \notin EF$), then we take $A = X$ and for B we choose the vertex of P from this triangle, which is the farthest from EF, but different from X. Finally, if $X \in EF$, then we take $A = X$ and as B we choose a suitable vertex of P, which is on the same side of EF as the polygon P itself in a small neighborhood of the point X (there is such a vertex that is visible from X, that is, that the interior points AB lie inside P).

Criteria. If a combinatorial construction is given, but the technical fact about the possibility of obtaining equal areas is not proven, then 5 points are given (a proof like "you can take an arc of the boundary of the polygon and continuously deform it into another, at some point the area will be as needed" is sufficient).